# The Simplex and Barycentric Coordinates 

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## 1 Barycentric Coordinates

Given $n$ points in a space $p_{1}, p_{2}, \ldots, p_{n}$, an $n-1$ dimensional simplex is the set of points

$$
\left\{p: p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\ldots+\lambda_{n} p_{n}\right\}
$$

where each $\lambda_{i}$ is between 0 and 1 , and they sum to 1 . Such a set of points is called an $n-1$ dimensional simplex. The set of $\lambda_{i}$ for a point $p$ are called the barycentric coordinates of the point, because if the coordinates are considered mass points at the vertices $p_{i}$, then $p$ is the center of mass.

Consider the one dimensional case defined by points $p_{1}$ and $p_{2}$. Assume $\lambda_{1}+\lambda_{2}=1$. Then

$$
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}
$$

is a point on the line through the points $p_{1}$ and $p_{2}$. And if $0<\lambda_{1}<1$ and $0<\lambda_{2}<1$ then $p$ is between $p_{1}$ and $p_{2}$. To prove this we write

$$
\begin{aligned}
& p=\lambda_{1} p_{1}+\lambda_{2} p_{2} \\
= & \left(1-\lambda_{2}\right) p_{1}+\lambda_{2} p_{2} \\
= & p_{1}+\lambda_{2}\left(\left(p_{2}-p_{1}\right) .\right.
\end{aligned}
$$

So $P$ is the sum of vectors $p_{1}$ and a multiple of $p_{2}-p_{1}$, and so lies on the line through $p_{1}$ and $p_{2}$. If $0<\lambda_{2}<1$ then clearly $p$ is between $p_{1}$ and $p_{2}$.

Now consider the case of three points $p_{1}, p_{2}, p_{3}$. If $\lambda_{1}+\lambda_{2}+\lambda_{3}=1$, then the set of points

$$
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3}
$$

are the points of the plane through $p_{1}, p_{2}, p_{3}$, (assuming these points are not collinear). Further, if each of the $\lambda_{i}$ are between 0 and 1 , then we get an interior point of the 2-dimensional simplex (triangle). We may prove this by writing

$$
\begin{gathered}
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3} \\
=\left(1-\lambda_{2}-\lambda_{3}\right) p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3} \\
=p_{1}+\lambda_{2}\left(\left(p_{2}-p_{1}\right)+\lambda_{3}\left(\left(p_{3}-p_{1}\right) .\right.\right.
\end{gathered}
$$

So $p$ is in the plane spanned by the two vectors $p_{2}-p_{1}, p_{3}-p_{1}$ from the origin $p_{1}$.

Now suppose $0<\lambda_{i}<1$ for $i=1,2,3$. Then let

$$
q=\lambda_{1} p_{1}+\lambda_{1} p_{2}
$$

and

$$
r=\frac{q}{\lambda_{1}+\lambda_{2}}=\frac{\lambda_{1}}{\lambda_{1}+\lambda_{2}} p_{1}+\frac{\lambda_{2}}{\lambda_{1}+\lambda_{2}} p_{2} .
$$

We have

$$
\frac{\lambda_{1}}{\lambda_{1}+\lambda_{2}}+\frac{\lambda_{2}}{\lambda_{1}+\lambda_{2}}=1
$$

So $r$ is between $p_{1}$ and $p_{2}$, that is, on edge $p_{1} p_{2}$. Then

$$
p=\left(\lambda_{1}+\lambda_{2}\right) r+\lambda_{3} p_{3} .
$$

So $p$ is between $r$ and $p_{3}$, thus interior to the 2 -simplex (triangle).
Proposition. Barycentric coordinates are unique.
proof. Suppose

$$
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3}=\lambda_{1}^{\prime} p_{1}+\lambda_{2}^{\prime} p_{2}+\lambda_{3}^{\prime} p_{3}
$$

Then

$$
p=p_{1}+\lambda_{2}\left(p_{2}-p_{1}\right)+\lambda_{3}\left(p_{3}-p_{1}\right)
$$

and

$$
p=p_{1}+\lambda_{2}^{\prime}\left(p_{2}-p_{1}\right)+\lambda_{3}^{\prime}\left(p_{3}-p_{1}\right) .
$$

So

$$
0=\left(\lambda_{2}-\lambda_{2}^{\prime}\right)\left(p_{2}-p_{1}\right)+\left(\lambda_{3}-\lambda_{3}^{\prime}\right)\left(p_{3}-p_{1}\right)
$$

If the coefficients are not zero, then $\left(p_{2}-p_{1}\right)$ and $\left(p_{3}-p_{1}\right)$ are linearly dependent, and $p_{1}, p_{2}, p_{3}$ are collinear. Hence the unprimed and primed coordinates are equal.

Now suppose a triangle $p_{1}, p_{2}, p_{3}$ is projected to the $x y$ plane with a transformation $T . T$ is linear so if

$$
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3}
$$

then

$$
T p=\lambda_{1} T p_{1}+\lambda_{2} T p_{2}+\lambda_{3} T p_{3}
$$

By uniqueness $p$ and $T p$ have the same barycentric coordinates. Hence the barycentric coordinates can be computed on the projected image, provided the projected points are still collinear. Then using these coordinates with the original points $p_{1} p_{2}, p_{3}$ we can find a point in the triangle, in the original simplex.

This can be used to to order triangles back to front in the $z$ direction for graphics drawing.

Suppose we are given an $n$-simplex with vertices $v_{0}, v_{1}, v_{2}, \ldots, v_{n}$. The barycentric coordinates of a point $p$ sum to one. If the coordinates satisfy

$$
0<\lambda_{i}<1
$$

then the point is an interior point of the simplex. If any coordinate is negative, then the point is exterior to the simplex. If

$$
0 \leq \lambda_{i} \leq 1
$$

then the point is in the interior or on the boundary of the simplex. In the case

$$
0 \leq \lambda_{i} \leq 1
$$

when a coordinate $\lambda_{j}=0$, the point is on the boundary of the simplex opposite the vertex $p_{j}$.

To find the barycentric coordinates we may select an arbitrary vertex, say $p_{n}$, and solve the linear system

$$
\sum_{i=0}^{n-1} \lambda_{i}\left(p_{i}-p_{n}\right)=p-p_{n}
$$

for $\lambda_{0}, \ldots, \lambda_{n-1}$. Since the barycentric coordinates sum to 1 , this also determines $\lambda_{n}$.

Let us apply this to the problem of determining that a point is in a triangle of the plane. Suppose we are given the triangle vertices

$$
\begin{aligned}
& p_{1}=(1,2), \\
& p_{2}=(1,3), \\
& p_{3}=(2,3)
\end{aligned}
$$

and wish to determine if $p=(1.5,2.6)$ is in the triangle. Our linear system is

$$
\left[\begin{array}{ll}
(1-2) & (1-2) \\
(2-3) & (3-3)
\end{array}\right]\left[\begin{array}{l}
\lambda_{1} \\
\lambda_{2}
\end{array}\right]=\left[\begin{array}{l}
(1.5-2) \\
(2.6-3)
\end{array}\right]
$$

The solution is

$$
\lambda_{1}=.1, \lambda_{2}=.4
$$

Then we compute $\lambda_{3}=.5$. Therefore, because all coordinates are between 0 and 1 , the point is in the triangle.

The general computation to determine an interior point, requires 11 additions or subtractions, 6 multiplications, 2 divisions, and 3 comparisons. The computation may be done as follows.

Let $a_{11}=x_{1}-x_{3}, a_{21}=y_{1}-y_{3}, a_{12}=x_{2}-x_{3}, a_{22}=y_{2}-y_{3}$, and $b_{1}=$ $x-x_{3}, b_{2}=y-y_{3}$. Then letting $D$ be the determinant

$$
D=a_{11} a_{22}-a_{21} a_{12}
$$

we have

$$
\begin{aligned}
& \lambda_{1}=\frac{b_{1} a_{22}-b_{2} a_{12}}{D} \\
& \lambda_{2}=\frac{a_{11} b_{2}-a_{21} b_{1}}{D}
\end{aligned}
$$

and

$$
\lambda_{3}=1-\lambda_{1}-\lambda_{2}
$$

See the C procedure bary2 (There is also a Fortran version).

```
//c+ bary2 barycentric coordinates of a point in the plane
int bary2(double* p,double* p1,double* p2,double* p3,double* lambda){
    double d,a11,a12,a21,a22,b1,b2;
    int i;
    a11=p1[0]-p3[0];
    a21=p1[1]-p3[1];
    a12=p2[0]-p3[0];
    a22=p2[1]-p3[1];
    b1=p[0]-p3[0];
    b2=p[1]-p3[1];
    d=a11*a22-a21*a12;
    if(d == 0.){
    return(0);
    }
    lambda[0]=(b1*a22 - b2*a12)/d;
    lambda[1]=(a11*b2-a21*b1)/d;
    lambda[2]=1.-lambda[0] - lambda[1];
    for(i=0;i<3;i++){
    if((lambda[i] <= - EPSILON) || (lambda[i] >= 1.+ EPSILON)){
        return(0);
    }
}
    return(1);
}
```

Let us now consider the problem of computing the barycentric coordinates of a point $P$ with respect to a triangle in space with vertices $P_{1}, P_{2} P_{3}$. Let

$$
\begin{aligned}
A_{1} & =P_{1}-P_{3} \\
A_{2} & =P_{2}-P_{3} \\
A & =P-P_{3}
\end{aligned}
$$

We shall set up a system of orthogonal vectors $U_{1}, U_{2}, U_{3}$ Let

$$
\begin{aligned}
& B_{3}=A_{1} \times A_{2}, \\
& B_{2}=B_{3} \times A_{1} .
\end{aligned}
$$

Let

$$
\begin{gathered}
U_{1}=\frac{A_{1}}{\left\|A_{1}\right\|}, \\
U_{2}=\frac{B_{2}}{\left\|B_{2}\right\|} \\
U_{3}=\frac{B_{3}}{\left\|B_{3}\right\|}=U_{1} \times U_{2}
\end{gathered}
$$

Using the "Back Minus Cab" rule, we have

$$
B_{2}=-\left(A_{1} \cdot A_{2}\right) A_{1}+\left(A_{1} \cdot A_{1}\right) A_{2}
$$

Let $P^{\prime}$ be the projection of $P$ to the plane of the triangle (which is $P$ if $P$ is already in that plane). Then we have

$$
\begin{gathered}
P^{\prime}=\left(A \cdot U_{1}\right) U_{1}+\left(A \cdot U_{2}\right) U_{2} \\
P_{1}=\left(A_{1} \cdot U_{1}\right) U_{1}+\left(A_{1} \cdot U_{2}\right) U_{2} \\
P_{2}=\left(A_{2} \cdot U_{1}\right) U_{1}+\left(A_{2} \cdot U_{2}\right) U_{2}
\end{gathered}
$$

Then we have the two dimensional case given above where the points $P, P_{1}, P_{2}, P_{3}$ have coordinates with respect to $U_{1}, U_{2}$

$$
\begin{aligned}
x & =A \cdot U_{1} \\
y & =A \cdot U_{2} \\
x_{1} & =A_{1} \cdot U_{1} \\
y_{1} & =A_{1} \cdot U_{2} \\
x_{2} & =A_{2} \cdot U_{1} \\
y_{2} & =A_{2} \cdot U_{2}
\end{aligned}
$$

and

$$
\begin{aligned}
x_{3} & =0 \\
y_{3} & =0
\end{aligned}
$$

The Fortran subroutine baryt implements this calculation (there is also C version.)

```
c+ baryt barycentric coordinates of a point relative to a triangle in space
    subroutine baryt(p,p1,p2,p3,b)
c input:
c p 3d point
c p1,p2,p3 3d points of triangle
c output:
c b barycentric coordinates of the projection
of p to the plane of the triangle.
projection(p)=b(1)*p1+b(2)*p2+b(3)*p3
c b(1)+b(2)+b(3) = 1
c the projection is outside the triangle if
c and only if some coordinate is negative
c Reference: Computer Graphics and Geometry, Jim Emery graphic.tex
    implicit real*8(a-h,o-z)
    dimension p(*),p1(*),p2(*),p3(*),b(*)
    dimension a(3),a1(3),a2(3),b1(3),b2(3),u1(3),u2(3)
    do i=1,3
        a1(i)=p1(i)-p3(i)
        a2(i)=p2(i)-p3(i)
        a(i)=p(i)-p3(i)
    enddo
    c1=-dotpr(a1,a2)
    c2=dotpr (a1,a1)
    do i=1,3
        b2(i)=c1*a1(i)+c2*a2(i)
    enddo
    c3=dotpr (b2,b2)
    do i=1,3
        u1(i)=a1(i)/sqrt(c2)
        u2(i)=b2(i)/sqrt(c3)
    enddo
    x=dotpr(u1,a)
    y=dotpr(u2,a)
    x1=dotpr(u1,a1)
    y1=dotpr(u2,a1)
    x2=dotpr(u1,a2)
    y2=dotpr(u2,a2)
    d=x1*y2-y1*x2
    b(1)=(x*y2-y*x2)/d
    b(2)=(y*x1-x*y1)/d
    b(3)=1.-b(1)-b(2)
    return
    end
```


### 1.1 Center of Mass

Given $n$ vectors in Euclidean Space, $p_{1}, \ldots, p_{n}$, if masses $\lambda_{1}, \ldots, \lambda_{n}$ are placed at these vectors, and

$$
M=\sum_{i=1}^{n} \lambda_{i}
$$

then

$$
p=\frac{1}{M} \sum_{i=1}^{n} \lambda_{i} p_{i}
$$

is the center of mass of these $n$ mass points, because this is the definition of the center of mass. If masses sum to 1 , then they are the normalized barycentric coordinates of the point $p$.

### 1.2 Triangles and Area Coordinates

Barycentric coordinates can also be called area coordinates or areal coordinates. The barycentric coordinates of a point $P$ inside a triangle divide the triangle into three internal triangles. It turns out that the barycentric coordinate $\lambda_{i}$ for vertex $P_{i}$ is the area of the internal triangle opposite $P_{i}$ divided by the area of the original triangle. A proof of this given below and is given in Coxeter p216, which relies on a theorem that the area of a triangle formed from drawing two lines from a vertex to the opposite side of a triangle is proportional to the length of the segment formed by the intersection of the two lines on the opposide side of the original triangle.

So first consider the barycentric coordinates of a point $\mathbf{P}$ on a line segment with end points $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$. Barycentric coordinates are homogenious coordinates, so only the ratios of the coordinates are completely specified. The point $\mathbf{P}$ is specified by

$$
\left(m_{1}+m_{2}\right) \mathbf{P}=m_{1} \mathbf{P}_{1}+m_{2} \mathbf{P}_{2},
$$

where $m_{1}, m_{2}$ are the barycentric coordinates written here as the masses at the two vertices. So $\mathbf{P}$ is the center of gravity of the masses. We can rewrite this as

$$
0=m_{1}\left(\mathbf{P}_{1}-\mathbf{P}\right)+m_{2}\left(\mathbf{P}_{2}-\mathbf{P}\right)
$$

so

$$
\left.m_{1}\left\|\mathbf{P}_{1}-\mathbf{P}\right\|=m_{2} \| \mathbf{P}_{2}-\mathbf{P}\right) \|
$$

which expresses the fact that the distances from an endpoint to the center of mass are inversly proportional to the masses. That is

$$
\frac{m_{1}}{m_{2}}=\frac{d_{2}}{d_{1}}
$$

where $d_{1}, d_{2}$ are the distances. This is the mechanical condition for equilibrium of the beam namely that the net torque is zero.

Now consider a triangle with vertices $\mathbf{P}_{1}, \mathbf{P}_{2}$, and $\mathbf{P}_{3}$. Then we have

$$
\left(m_{1}+m_{2}+m_{3}\right) \mathbf{P}=m_{1} \mathbf{P}_{1}+m_{2} \mathbf{P}_{2}+m_{3} \mathbf{P}_{3}
$$

where $m_{1}, m_{2}, m_{3}$ are the barycentric coordinates of a point $\mathbf{P}$ in the triangle. Then

$$
\left(m_{1}+m_{2}+m_{3}\right) \mathbf{P}=\left(m_{1}+m_{2}\right) \mathbf{Q}+m_{3} \mathbf{P}_{3}
$$

where

$$
\mathbf{Q}=\frac{m_{1} \mathbf{P}_{1}+m_{2} \mathbf{P}_{2}}{m_{1}+m_{2}}
$$

is the barycenter of the masses at $\mathbf{P}_{1}$ and $\mathbf{P}_{2}$. Notice that if $\mathbf{A}$ and $\mathbf{B}$ are two vectors such that

$$
\mathbf{C}=\alpha \mathbf{A}+\beta \mathbf{B}
$$

where $\alpha+\beta=1$, then

$$
\mathbf{C}=(1-\beta) \mathbf{A}+\beta \mathbf{B}=\mathbf{A}+\beta(\mathbf{B}-\mathbf{A})
$$

is on the line joining points $\mathbf{A}$ and $\mathbf{B}$. Therefore $\mathbf{Q}$ is on the line segment $\mathbf{P}_{1} \mathbf{P}_{2}$ and $\mathbf{P}$ is on the line segment $\mathbf{Q} \mathbf{P}_{3}$. And we now see that the following ratios are equal

$$
\frac{m_{1}}{m_{2}}=\frac{d_{2}}{d_{1}}=\frac{\mathbf{Q P}_{2}}{\mathbf{P}_{1} \mathbf{Q}}=\frac{\mathbf{Q P}_{2} \mathbf{P}_{3}}{\mathbf{P}_{1} \mathbf{Q P}}
$$

The last equality for the ratio of triangle areas follows because these two triangles $\mathbf{Q P}_{2} \mathbf{P}_{3}$ and $\mathbf{P}_{1} \mathbf{Q P} \mathbf{P}_{3}$ have the same height, and bases $d_{2}$ and $d_{1}$ respectively. Continuing

$$
\frac{m_{1}}{m_{2}}=\frac{\mathbf{Q P}_{2} \mathbf{P}_{3}}{\mathbf{P}_{1} \mathbf{Q P} P_{3}}=\frac{\mathbf{Q P}_{2} \mathbf{P}}{\mathbf{P}_{1} \mathbf{Q P}}=\frac{\left(\mathbf{Q P}_{2} \mathbf{P}_{3}-\mathbf{Q P}_{2} \mathbf{P}\right)}{\left(\mathbf{P}_{1} \mathbf{Q} \mathbf{P}_{3}-\mathbf{P}_{1} \mathbf{Q P}\right)}=\frac{\mathbf{P}_{2} \mathbf{P P}_{3}}{\mathbf{P}_{1} \mathbf{P P}_{3}}
$$

which shows that the barycentric coordinate for $\mathbf{P}_{i}$ is the area of the triangle opposite $\mathbf{P}_{i}$. We get a normalized coordinate by dividing by the area of the whole original triangle. Thus as $\mathbf{P}$ approaches $\mathbf{P}_{i}$ the coordinate $\lambda_{i}$ approaches one, and the other two coordinates approach zero. We have proved the following proposition.
Proposition. Let $\mathbf{P}$ a point, and $\mathbf{P}_{1}, \mathbf{P}_{2}, \mathbf{P}_{3}$ the vertices of a triangle $\mathbf{P}_{1} \mathbf{P}_{2} \mathbf{P}_{3}$. This triangle has clockwise orientation. Let $a$ be the area of $\mathbf{P}_{1} \mathbf{P}_{2} \mathbf{P}_{3}, a_{1}$ the area of $\mathbf{P} \mathbf{P}_{2} \mathbf{P}_{3}, a_{2}$ the area of $\mathbf{P} \mathbf{P}_{3} \mathbf{P}_{1}$, and $a_{3}$ the area of $\mathbf{P P}_{1} \mathbf{P}_{2}$, where each area is positive if the vertices go clockwise, and negative otherwise. Then

$$
\lambda_{i}=\frac{a_{i}}{a},
$$

are the normalized baracentric coordinates of $\mathbf{P}$ with respect to the triangle $\mathbf{P}_{1} \mathbf{P}_{2} \mathbf{P}_{3}$. That is

$$
\mathbf{P}=\lambda_{1} \mathbf{P}_{1}+\lambda_{2} \mathbf{P}_{2}+\lambda_{3} \mathbf{P}_{3}
$$

and

$$
\lambda_{1}+\lambda_{2}+\lambda_{3}=1 .
$$

### 1.3 Barycentric Coordinates and the Convex Hull

The convex hull $H(A)$ of a set of points $A$ is the smallest set that contains all line segments connecting pairs of points of $A$.

### 1.4 The Simplex and the Simplicial Complex

## 2 Algebraic Topology

## 3 Barycentric Coordinates For Triangular Finite Elements (fecrrnt.tex)

Note. This document was not published in $\mathrm{T}_{\mathrm{E}} \mathrm{X}$, and only the first part has been fully converted here. It was written a word processing language of my design called "script" and was written in Pascal. Spaces were left for mathematical symbols, which were entered by hand, as was typical in those ancient days. The fecurrent finite element program itself was written in Fortran.

Barycentric coordinates are useful in triangular finite elements.
Let $\alpha_{1}, \alpha_{2}$, and $\alpha_{3}$ be points in a two dimensional projective space. Let $\alpha$ be an arbitrary point in the space. Then $\alpha$ is a one dimensional subspace of some three dimensional vector space. We wish to assign coordinates to points of the projective space. Suppose $p_{1}, p_{2}, p_{3}$ are representative vectors of the three basic projective points $\alpha_{1}, \alpha_{2}$, and $\alpha_{3}$ respectively. We suppose that the vectors are linearly independent. Let $p$ be in $\alpha$. Then $p$ has a
set of coordinates with respect to the three linearly independent vectors. However, these coordinates will vary depending upon how we select $p_{1}, p_{2}$, $p_{3}$ and $p$ from the respective one dimensional subspaces. However, suppose we identify a fourth projective point $\alpha_{u}$ that we take to have unit coordinates $\lambda_{1}=1, \lambda_{2}=1, \lambda_{3}=1$. Now because a projective point is a one dimensional subspace, any constant multiple of these coordinates will have to represent the same projective point. That is the unit point will also have coordinates, say, $\lambda_{1}=5, \lambda_{2}=5, \lambda_{3}=5$. The unit point forces us to select a certain set of basis vectors. These are vectors $p_{1}, p_{2}, p_{3}$ so that if $p_{u} \in \alpha_{u}$, then

$$
p_{u}=c p_{1}+c p_{2}+c p_{3},
$$

for some number $c$. Then $p_{1}, p_{2}, p_{3}$ are well defined, (except they could all be scaled by the same number. Then the coordinates of any $\alpha$ are well defined by, if $p \in \alpha$, then

$$
p=\lambda_{1} p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3} .
$$

And so these are the well defined projective coordinates defined by the projective space reference points $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and the unit point $\alpha_{u}$.

We shall introduce a special projective coordinate system. Let $p_{1}, p_{2}$, and $p_{3}$ be the vertices of a triangle. The baracentric coordinate system is a projective coordinate system with reference points $\left[p_{1}\right],\left[p_{2}\right],\left[p_{3}\right]$ and unit point

$$
\left(p_{1}+p_{2}+p_{3}\right) / 3
$$

$\left(p_{1}+p_{2}+p_{3}\right) / 3$ is the center of gravity (the barycenter) of the triangle. If $p$ is an arbitrary point, then

$$
p=\lambda p_{1}+\lambda_{2} p_{2}+\lambda_{3} p_{3}
$$

We may assume that the coordinates are scaled so that

$$
\lambda_{1}+\lambda_{2}+\lambda_{3}=1
$$

The barycentric coordinates are also vector space coordinates in the three dimensional vector space with basis vectors $p_{1}, p_{2}, p_{3}$. Here we shall think of them in that way. We will deduce some relationships for the coordinates that hold for permutations of the basis vectors $p_{1}, p_{2}, p_{3}$. So let $\sigma$ be a permutation of $1,2,3$. (see fecrrnt.tex) for a continuation of this.

Then

$$
P-P=(P-P)+(P-P)
$$

SO

$$
=
$$

$$
y-y y-y y-y
$$

Let $\$ \mathrm{D} \$$ be the determinant of this matrix. Then

|  | 0 |  | 0 | 1 | 1 | 1 | 1 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $x$ | $-x$ | $x$ | $-x$ | $x$ | $x$ | $x$ | $x$ |
| $y$ | $-y$ | $y$ | $-y$ | $y$ | $y$ | $y$ | $y$ |

$=\operatorname{sign}\left(\begin{array}{lll}1 & 1 & 1 \\ x & x & x \\ y & y & y\end{array}\right.$
.sp
Using Cramer's rule we find
$\left.=\left(\begin{array}{ll}x & y-x\end{array} \quad y\right)+\left(\begin{array}{ll}y & -y\end{array}\right) x+(x-x \quad) y\right) / D$
. sp
.sp
$a=x \quad y-x \quad y$
.sp
b $=\mathrm{y}-\mathrm{y}$
.sp
$c=x \quad-x$
.sp
Then
.sp
$=(1 / D)(a+b x+c y)$
.sp
Note that is well defined: Suppose '1 = 1, and $' 2=2$. Then $\operatorname{sign}(')=-\operatorname{sign}()$ and
.sp
$\mathrm{a}=-\mathrm{a}, \mathrm{b}=-\mathrm{b}, \mathrm{c}=-\mathrm{c}$
.s
so =
.sp
We introduce 3 even permutations
.sp

| 123 | 123 | 123 |
| :---: | :---: | :---: |
|  |  |  |
| 123 | 231 | 312 |

.sp
Then
. SP
$(x, y)=(1 / D)(a+b x+c y)$

```
.sp
where
.sp
a = x y - x y
.sp
b = y - y
.sp
c = x - x
.sp
a = x y - x y
.sp
b = y - y
.sp
c = x - x
.sp
a = x y - x y
.sp
b = y - y
.sp
c = x - x
.sp
and
.sp
D = llll
y y y
.sp
The magnitude of D is twice the area of the triangle.
The interpolation function in the triangle is
.sp
u = u + u + u
.sp
.sp
grad}(u)=(1/D)( b u + b u c + c u c )
.sp
An outward normal vector to an edge opposite vertex i is
.sp
n = -(sign(D)/L )
.sp
where L is the edge length.
.sp
The normal derivative at the edge opposite vertex i is
.sp
    u/ n =grad(u) n
.sp
    =-(\operatorname{sign}(D)/DL)}\mp@subsup{}{j=1}{3}(bb+cc)
    .sp
The nodal flow at vertex i is
.sp}
        u/ n dL
```

```
.sp
where the integration is done on the two half edges
meeting at vertex i. Then
.sp
            3
N =(1/(2 D ) 
.sp
Also because u is harmonic
.sp
N}
    A
.sp
.fo
where A is the line connecting the midpoints of the
edges incident at nodei, and so can be thought
of as the flow from the node.
.SP
.ce
THE LINEAR NODAL EQUATIONS
.SP
The functional F given in (16) is to be minimized over
the finite dimensional space of linear interpolation
functions. Suppose there are m elements and n nodes.
Let E be the jth element. Define a functional F on
the jth element by
.nf
.sp
F}(u)=@(u/x)+(u/y)-2 uh
E
E
.sp
We have
.sp
F(u) = m
.sp
Let i = n(j,k) be the global index of node k of element j.
We have
.sp
F/u= 2 E @ u/x /u (u/x) + u/y / u(u/y)
.sp 2
        - 2 u/ u h
        E
.sp
Now locally we have
    .sp
    F/u = F / u
.sp
u/ x = (1/D)@b u + b u + b u
.sp
u/ y = (1/D)@c u + c u + c u
.sp
    / u ( u/ x) = b /D
.sp
    / u ( u/ y) = c /D
```

```
.sp
    u/ u =
.sp
.sp
.sp
F / u =
.sp /D
@(bb+cc)u+(b b +cc)u+(b b + c c ) u
.sp
- 2 E
.sp
=(2 /D)(D /2) (b b + c c )u - 2
q=1
.sp
Define the jth element stiffness matrix by
. .sp}=(/D)(bb+cc
.sp
We denote by I the integral
. . Sp}
h
which depends only on the given boundary normal derivative \(h\).
Then locally
.sp
F/u = 
Note that if E = 0, then I = 0, and also
.sp
3
    K u = N
q=1
.sp
which is the local nodal flow. Now let i be a global label.
Then
.sp
                    n
F/u = N=1 n(j,k)=i K u - 2 n n(j,k)=i
                n(j,q)=l
.sp
Let
.sp
```



```
        n(j,q)=l
.sp
K}\mathrm{ is the global stiffness matrix. Then
.sp
f/u = K u - 2I
(**)
```

```
                l=1
.SP
note that
.sp
            n
(1/2) K u
    l=1
.sp
.fo
is the sum of the local nodal flows in the triangles that
are adjacent to node i. It is the sum of the current
flow out of the polygon, which is formed by the lines
connecting the midpoints of the edges incident at node i.
.sp 10
INTERNAL NODE BOUNDARY NODE
.SP
The boundary conditions consist of specified voltage
values and specified normal derivatives given by the
functions g}\mathrm{ and h respectively. g specifies the values
at certain boundary nodes while h specifies values of
I at the other boundary nodes. Of course, these
nodal boundary values will give only approximations
to the functions g}\mathrm{ and h. A necessary condition
for the functional to have a minimum is that
.sp
    F/ u = 0
.sp
for each unspecified node i. From (**) we will
then have p linear equations in p unknowns where
p is the number of unspecified nodes. The
solution u satisfies the natural boundary condition
in the sense that if i is a node that is in
then
.sp
.nf
C u/n dL = }\mp@subsup{\mp@code{l=1}}{n}{k}u=2I=2 hdL
.sp }1
.fo
Then u/ n is an approximation to h in a neighborhood of
each
node.
.sp
When we include the unspecified nodes we get the system
.sp
.nf
    u I
K - 2 = 0
    u I
.sp
.fo
which is a sytem of n equations in the n unknowns
consisting of the p unknown voltage values and the
n-p unknown current values. And we see that these
later values will be the nodal flows of the
solution function at the boundary nodes that are
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## 4 Bibliography

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